JOURNAL OF MULTIVARIATE ANALYSIS 38, 213–232 (1991)

Expansions for the Multivariate Chi-Square Distribution

T. ROYEN

Fachhochschule Rheinland-Pfalz, Abteilung Bingen, Eschborn, Germany

Communicated by C. R. Rao

Three classes of expansions for the distribution function of the $\chi_k^2(d, R)$-distribution are given, where k denotes the dimension, d the degree of freedom, and R the "accompanying correlation matrix." The first class generalizes the orthogonal series with generalized Laguerre polynomials, originally given by Krishnamoorthy and Parthasarathy [12]. The second class contains always absolutely convergent representations of the distribution function by univariate chi-square distributions and the third class provides also the probabilities for any unbounded rectangular regions. In particular, simple formulas are given for the three-variate case including singular correlation matrices R, which simplify the computation of third order Bonferroni inequalities, e.g., for the tail probabilities of $\max\{\chi_i^2 \mid 1 \leqslant i \leqslant k\}$ $(k > 3)$. © 1991 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper we use the following notations: The spectral norm of any $k \times k$-matrix $A = (a_{ij})$ is denoted by $\|A\|$ and the norm $\max\{\sum_{j=1}^k |a_{ij}|\}$ by $\|A\|_1$. We write $|A|$ for the determinant of A and $|\det(A)|$ for its absolute value. $\dot{A}$ is defined as the matrix $A - \mathrm{Diag}(a_{11}, ..., a_{kk})$ and $A > 0$ means positive definiteness of A. A unit matrix is always denoted by I. The letter μ refers to the Lebesgue measure and $\mathscr{L}^2$ stands for the Hilbert space $\mathscr{L}^2(\mathbb{R}^k, \mu)$. The notation $\sum_{(N)}$ means a summation over all partitions of a non-negative integer $N = \sum N_i$ (or $\sum N_{ij}$) with non-negative integers N_i, $i = 1, ..., k$ (or N_{ij}, $1 \leqslant i < j \leqslant k$). Formulas from the handbook of mathematical functions by Abramowitz and Stegun [1] are cited by "A. S." and their number.

Let $R = (r_{ij})$ denote any non-singular $k \times k$-correlation matrix, $Y_1, ..., Y_d$ independent $N(0, R)$-distributed random vectors and Y the $k \times d$-matrix

Received March 20, 1990; revised December 18, 1990.

AMS 1980 subject classifications: 62H10, 62E15.

Key words and phrases: multivariate chi-square distribution, multivariate gamma distribution, multivariate Rayleigh distribution.

683/38/2-4

0047-259X/91 \$3.00

with the columns Y_j. We want to expand the joint distribution function (d.f.) $G_{k,d,R}$ of the squared Euclidean norms $X_i = \chi_i^2$ of the row vectors of Y. This is the d.f. of a k-variate gamma distribution of order $p = d/2$ in the sense of Krishnamoorthy and Parthasarathy [12] with the accompanying correlation matrix R. For the multivariate gamma distribution see also Krishnaiah and Rao [11], Johnson and Kotz [7], Krishnaiah [10] and for the multivariate Rayleigh distribution (i.e., the joint distribution of the $\sqrt{X_i}$) see Miller *et al.* [14], Blumenson and Miller [3], and Jensen [6].

Apart from some special cases no "simple formulas" are known for the $\chi_k^2(d, R)$-density $g_{k,d,R}$. A comparatively simple case arises if R is "one-factorial" (cf. end of Section 2.3).

Let G_{p+n} denote the d.f. of the gamma distribution of order $p+n$ and $G_{p+n}^{(n)}$ its nth derivative ($p > 0$, $n \in \mathbb{N}_0$). For the bivariate case with $R = \left(\begin{smallmatrix} 1 & r \\ r & 1 \end{smallmatrix}\right)$ the two series

$$G_{2,2p,R}(x_1, x_2)$$

$$= \frac{1}{\Gamma(p)} \sum_{n=0}^{\infty} (\Gamma(p+n) r^{2n}/n!) G_{p+n}^{(n)}(x_1/2) G_{p+n}^{(n)}(x_2/2) \tag{1.1a}$$

$$= \frac{(1-r^2)^p}{\Gamma(p)} \sum_{n=0}^{\infty} (\Gamma(p+n) r^{2n}/n!) G_{p+n}\left(\frac{x_1/2}{1-r^2}\right) G_{p+n}\left(\frac{x_2/2}{1-r^2}\right) \tag{1.1b}$$

are well known (e.g., Johnson and Kotz [7]).

In the following section three classes of expansions for $G_{k,2p,R}$ are given. The first one contains generalizations of (1.1a) including the series given by Krishnamoorthy and Parthasarathy [12]. The second class contains generalizations of (1.1b) including the expansions which could also be derived by convolutions of the series of Ǎndel [2] for multivariate normal probabilities of symmetrical rectangles $\bigotimes_{j=1}^{k} (-b_j, b_j)$. This class provides always absolutely convergent representations of $G_{k,d,R}$. Finally, a third class is derived which always gives absolutely convergent expansions also for the probabilities of any unbounded rectangular regions $\mathscr{R} = \bigotimes_{j=1}^{k} (a_j, b_j)$ with $0 \leqslant a_j < b_j \leqslant \infty$.

In Section 3 special formulas for the three-variate case are given which simplify, e.g., the computation of third-order Bonferroni inequalities for the tail probabilities of $\max\{\chi_j^2 \mid 1 \leqslant j \leqslant k\}$ with $k > 3$.

In their paper [12] Krishnamoorthy and Parthasarathy derived an expansion of the multivariate gamma density based on generalized Laguerre polynomials and they showed its absolute convergence under the condition

$$\sum_{m=2}^{k} \sum_{|J|=m} |\det(\dot{R}_J)| < 1. \tag{1.2}$$

Here R_J is the submatrix of R with the row and column numbers of $J=\{j_1, ..., j_m\}$. Their series is an example of an expansion of a multivariate density g by functions orthogonal with respect to a weight function w (Lancaster [13]). In their paper w is the product of the univariate marginal gamma densities $g_p(x_j)=x_j^{p-1}\exp(-x_j)/\Gamma(p)$. To facilitate the integration of such a series only products of univariate probability densities $w_j(x_j)$ are used as weight functions in this paper. We assume that the supports of the w_j are some intervals S_j and that the support of g is a subset of the support S of w. If

$$f:=g/\sqrt{w}\in\mathscr{L}^2(S) \tag{1.3}$$

and if the sets $\{L_{j,n}\sqrt{w_j}\,|\,n\in\mathbf{N}_0\}$ form complete orthonormal systems of $\mathscr{L}^2(S_j)$ $(j=1, ..., k)$, then g is the $\mathscr{L}^1$-limit of a sequence (g_N) with

$$g_N(x)=w(x)\sum_{n=0}^{N}\sum_{(n)}c^*(n_1, ..., n_k)\prod_{j=1}^{k}L_{j,n_j}(x_j) \tag{1.4}$$

and the coefficients

$$c^*(n_1, ..., n_k)=\int_S\left(\prod_{j=1}^{k}L_{j,n_j}(x_j)\right)g(x)\,d\mu=E\left(\prod_{j=1}^{k}L_{j,n_j}(X_j)\right). \tag{1.5}$$

For any measurable subset $\mathscr{A}$ of S it holds with $f_N=g_N/\sqrt{w}$:

$$\begin{aligned}\left|\int_{\mathscr{A}}g\,d\mu-\int_{\mathscr{A}}g_N\,d\mu\right|&\leqslant\int_{\mathscr{A}}|f-f_N|\sqrt{w}\,d\mu\\&\leqslant\left(\int_{\mathscr{A}}w\,d\mu\right)^{1/2}\left(\int_{\mathscr{A}}|f-f_N|^2\,d\mu\right)^{1/2}\leqslant\|f-f_N\|_2\\&=\left(\sum_{n>N}\sum_{(n)}c^{*2}(n_1, ..., n_k)\right)^{1/2}\end{aligned} \tag{1.6}$$

which tends to zero for $N\to\infty$.

The condition (1.3), however, is rather restrictive if w is the product of the univariate marginal densities of g. For example, let g be a multivariate normal density proportional to $\exp(-\frac{1}{2}x'R^{-1}x)$ and w a weight function proportional to $\exp(-x'Dx)$ with $D=\mathrm{Diag}(d_1, ..., d_k)>0$. Obviously (1.3) holds iff $R^{-1}-D>0$. In particular, we have

$$R^{-1}-\tfrac{1}{2}I>0 \qquad \text{iff} \quad \|\dot{R}\|<1. \tag{1.7}$$

A similar behaviour of the multivariate gamma density $g=g_{k,d,R}$ is shown by the following theorem:

THEOREM 1.1. *Let D be a diagonal matrix with positive elements $d_1, ..., d_k$ and $f := g_{k,2p,R} \exp(\frac{1}{2}\sum_{j=1}^{k} d_j x_j)$ with $2p > 1$.*

(a) *If $R^{-1} - D > 0$ then $f \in \mathscr{L}^1 \cap \mathscr{L}^2$.*

(b) *If $R^{-1} - D \geq 0$ with at least one vanishing characteristic root then $f \notin \mathscr{L}^1 \cup \mathscr{L}^2$.*

Proof. (a) With $T = \mathrm{Diag}(t_1, ..., t_k)$ the Fourier transform (F.t.) of f is given by $\varphi(t) = |I - DR - 2iTR|^{-p}$. From $\varphi(0) = |R|^{-p}\, |R^{-1} - D|^{-p} < \infty$ it follows $f \in \mathscr{L}^1$.

Let (b_{ij}) be the matrix $(R^{-1} - D)^{-1} = B_1 B_0 B_1$ with $B_1 = \mathrm{Diag}(b_{11}^{1/2}, ..., b_{kk}^{1/2})$ and $U = \mathrm{Diag}(u_1, ..., u_k)$ with $u_j = -2ib_{jj}t_j/(1 - 2ib_{jj}t_j)$. Then we have $\varphi(t) = \varphi(0)\, |I - 2iTB|^{-p} = \varphi(0)(\prod_{j=1}^{k}(1 - 2ib_{jj}t_j))^{-p}$ $|I + U\dot{B}_0|^{-p}$. With $\omega_j := \exp(i2 \arctan(2b_{jj}t_j))$, $\Omega = \mathrm{Diag}(\omega_1, ..., \omega_k)$ and $U = \frac{1}{2}(I - \Omega)$ we find

$$|I + U\dot{B}_0| = |\tfrac{1}{2}(B_0 + I)|\, |I - (I - 2(B_0 + I)^{-1})\Omega|,$$

$$\|I - 2(B_0 + I)^{-1}\| < 1.$$

This entails $\varphi \in \mathscr{L}^2$ and therefore $f \in \mathscr{L}^2$.

(b) It is sufficient to consider a positive semi-definite matrix $R^{-1} - D$, which is obtained from a positive definite one by increasing only one element of D. Then $|R^{-1} - D| = 0$ implies that $|R^{-1} - D - 2iT| \simeq b't$ for $\|t\| \to 0$ with a certain vector $b \neq 0$ and it follows that $\varphi \notin \mathscr{L}^2$ and $f \notin \mathscr{L}^2$. ∎

Now let Z denote a $N(0, \Sigma)$-distributed random vector. An always absolutely convergent series for $\Pr\{Z \in \bigotimes_{j=1}^{k}(-b_j, b_j)\}$ was given by Ǎndel [2] (see also Šidák [19]). With the notations $R^{-1} = (r^{ij})$, $W = \mathrm{Diag}(\sqrt{r^{11}}, ..., \sqrt{r^{kk}})$, and the standardized inverse

$$Q = (q_{ij}) := (WRW)^{-1}, \tag{1.8}$$

we obtain from Ǎndel's expansion,

$$G_{k,1,R}(x_1, ..., x_k) = |Q|^{1/2} \left(\prod_{j=1}^{k} G_{1/2}(r^{jj}x_j/2) + \sum_{N=2}^{\infty} P_N \right) \tag{1.9}$$

with

$$P_N(x_1, ..., x_k) = (-2)^N \sum_{N_1 + \cdots + N_k = N} \left(\sum_{n_i = 2N_i} \prod_{i<j} (q_{ij}^{n_{ij}}/n_{ij}!) \right)$$
$$\times \prod_{j=1}^{k} (\Gamma(\tfrac{1}{2} + N_j)/\Gamma(\tfrac{1}{2}))\, G_{1/2+N_j}(r^{jj}x_j/2), \tag{1.10}$$

where the inner sum is extended over all partitions $N=\sum_{i<j} n_{ij}$ with $n_i := \sum_{j=1}^{k} n_{ij} = 2N_i$ $(i=1, ..., k,\ n_{ii} := 0,\ n_{ij} := n_{ji}$ for $j<i)$. We refer to (1.9) as to "Ăndel's series." Especially for $k=2$ this series simplifies to (1.1b) with $p=\frac{1}{2}$. The corresponding expansion for one-sided rectangles $\bigotimes_{j=1}^{k}(a_j, \infty)$ was considered by Moran [15] and found to be absolutely convergent for $\|\dot{Q}\|_1 < 1$.

2. The General Expansions

2.1. *The Three Types of Expansions*

The characteristic function (ch.f.) of $g_{k,2p,R}$ is given by

$$\psi = \psi_{k,2p,R}(t_1, ..., t_k) = |I - 2iRT|^{-p} \tag{2.1.1}$$

with $T = \mathrm{Diag}(t_1, ..., t_k)$. For any $t=(t_1, ..., t_k)' \in \mathbb{R}^k$ and any fixed positive numbers $d_1, ..., d_k$ we define

$$z_j := (1 - it_j/d_j)^{-1}, \qquad u_j := (-it_j/d_j) z_j = 1 - z_j. \tag{2.1.2}$$

The function $z_j = (1 - it_j/d_j)^{-1}$ transforms the upper half-plane $\mathrm{Im}(t_j) \geqslant 0$ onto the disc $\{z \in \mathbf{C} \mid |z - \frac{1}{2}| \leqslant \frac{1}{2}, z \neq 0\}$. Thus we have with

$$\varphi_j := 2 \arctan(t_j/d_j), \qquad \omega_j := \exp(i\varphi_j), \tag{2.1.3}$$

the relations

$$z_j = (1 + \omega_j)/2, \qquad u_j = (1 - \omega_j)/2. \tag{2.1.4}$$

The F.t. of $d_j g_{p+n}^{(m)}(d_j x_j)$, where $g_{p+n}^{(m)}(x)$ denotes the mth derivative of the gamma density $g_{p+n}(x)$ $(p>0,\ 0 \leqslant m \leqslant n \in \mathbf{N}_0)$, is given by

$$z_j^{p+n-m} u_j^m \qquad (\mathrm{Re}(z_j^{1/2}) > 0). \tag{2.1.5}$$

The generalized Laguerre polynomials $L_n^{(\alpha)}$ $(\alpha > -1)$ are defined by

$$\Gamma(\alpha + 1 + n)\, g_{\alpha+1+n}^{(n)}(x) = e^{-x} x^{\alpha} n!\, L_n^{(\alpha)}(x) \tag{2.1.6}$$

(A.S.22.11.6). We need the following bounds (A.S.22.14.13/14):

$$|g_{p+n}^{(n)}(x)| \leqslant \begin{cases} e^{-x/2} x^{p-1}/\Gamma(p), & p \geqslant 1 \\ e^{-x/2} x^{-1/2} (2n!/\Gamma(\frac{1}{2}+n) - 1/\sqrt{\pi}), & p = 1/2. \end{cases} \tag{2.1.7}$$

Furthermore, $z_j^p \omega_j^n = z_j^p (2z_j - 1)^n$ is the F.t. of $d_j h_{p,n}(d_j x_j)$ with

$$\begin{aligned} h_{p,n}(x) &= \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} 2^m g_{p+m}(x) \\ &= \binom{-p}{n}^{-1} g_p(x) L_n^{(p-1)}(2x), \end{aligned} \tag{2.1.8}$$

which is the derivative of

$$H_{p,n}(x) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} 2^m G_{p+m}(x). \tag{2.1.9}$$

From (2.1.2), (2.1.4), (2.1.5), and (2.1.6) we also obtain the recursion formulas

$$H_{p,n+1} = H_{p,n} - 2h_{p+1,n} = H_{p+1,n} - h_{p+1,n}. \tag{2.1.10}$$

The Fourier inversion formula implies

$$\begin{aligned} H_{p,n}(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} (\sin(tx)/t)(1-it)^{-p} \exp(in\, 2 \arctan(t))\, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\sin(x \tan(\varphi/2))/\sin(\varphi)) \\ &\quad \times ((1+\exp(i\varphi))/2)^p \exp(in\varphi)\, d\varphi. \end{aligned}$$

Thus for any fixed x the $H_{p,n}$ can be considered as the Fourier coefficients of a function of $\mathscr{L}^1(-\pi, \pi)$, which entails

$$\lim_{n \to \infty} H_{p,n}(x) = 0. \tag{2.1.11}$$

Now with any scale factors $d_j = w_j^2/2$, the ch.f. $\psi_0 := \prod_{j=1}^{k} z_j^p$, (2.1.2), (2.1.4), and $D = \mathrm{Diag}(d_1, ..., d_k)$, $W = \mathrm{Diag}(w_1, ..., w_k)$, $U = \mathrm{Diag}(u_1, ..., u_k)$, $Z = I - U$, and $\Omega = 2Z - I$, we find

$$\begin{aligned} |I - 2iRT| &= |I - iWRWTD^{-1}| = |I - iTD^{-1} + (I - WRW)iTD^{-1}| \\ &= \psi_0^{-1} |I - (I - WRW)U|, \end{aligned}$$

which leads to three representations of the ch.f. ψ in (2.1.1) of the general form

$$\psi = c\psi_0^p |I - CY|^{-p}, \qquad C = I - A \tag{2.1.12}$$

with

$$\begin{aligned} &Y = U, \quad A = WRW, &&c = 1 &&(2.1.12\text{a}) \\ &Y = Z, \quad A = (WRW)^{-1}, &&c = |A|^p &&(2.1.12\text{b}) \\ &Y = \Omega, \quad A = 2(I + WRW)^{-1}, &&c = |A|^p. &&(2.1.12\text{c}) \end{aligned}$$

By inversion of (2.1.12) we obtain the formal expansions

$$c \sum_{n=0}^{\infty} \sum_{(n)} c(n_1, ..., n_k) \prod_{j=1}^{k} (F_{p,n_j}(d_j b_j) - F_{p,n_j}(d_j a_j)) \tag{2.1.13}$$

for the probabilities $\Pr\{X \in \bigotimes_{j=1}^{k} (a_j, b_j)\}$ $(0 \leqslant a_j < b_j \leqslant \infty)$ with

$$F_{p,n} = \begin{cases} G_{p+n}^{(n)} & \text{(2.1.13a)} \\ G_{p+n} & \text{(2.1.13b)} \\ H_{p,n}, & \text{(2.1.13c)} \end{cases}$$

where the coefficients $c(n_1, ..., n_k)$ depend on p and C $(G_{p+n}(\infty) = 1$, $G_{p+n}^{(n)}(\infty) = 0$ for $n \geqslant 1$, and $H_{p,n}(\infty) = 1)$.

In (2.1.12c) the condition

$$\|C\| < 1 \tag{2.1.14}$$

holds for all R, $W > 0$. In (2.1.12a) or (2.1.12b) the condition $\|C\| < 1$ can be satisfied by a suitable diagonal matrix $D = \frac{1}{2} W^2$, since

$$R^{-1} - \tfrac{1}{2} W^2 > 0 \text{ is equivalent to } \|I - WRW\| < 1 \tag{2.1.15a}$$

and

$$R - \tfrac{1}{2} W^{-2} > 0 \text{ is equivalent to } \|I - (WRW)^{-1}\| < 1. \tag{2.1.15b}$$

The expansion

$$|I - CY|^{-p} = \left| \sum_{n=0}^{\infty} \binom{-p}{n} (-CY)^n \right| = \sum_{n=0}^{\infty} \hat{P}_n \tag{2.1.16}$$

with certain homogeneous polynomials

$$\hat{P}_n(y_1, ..., y_k) = \sum_{(n)} c(n_1, ..., n_k) \prod_{j=1}^{k} y_j^{n_j} \tag{2.1.17}$$

holds for $\|CY\| < 1$. For half-integers p this is justified because $\varphi(y) = |I - CY|^{-p}$ is uniquely defined on the domain $\{\|CY\| < 1\}$ by $\varphi(0) := 1$ and by continuity.

In (2.1.12) $\|Y\|$ is always bounded by 1. If ϑ is any number with $\|C\| < \vartheta < 1$ then $|\hat{P}_n| = \mathcal{O}_{n \to \infty}(\vartheta^n)$ with an $\mathcal{O}$-constant depending on ϑ, k, and p. This holds since the determinant in (2.1.16) is a sum of $k!$ products of k absolutely convergent series and the nth term of such a product series is absolutely bounded by $\|CY\|^n \sum_{(n)} \prod_{j=1}^k |\binom{-p}{n_j}| = \mathcal{O}(\vartheta^n)$. Substituting the variables y_j by $\exp(i\varphi_j)$ $(-\pi < \varphi_j < \pi)$ it follows that

$$(2\pi)^{-k} \int_{(-\pi,\pi)^k} |\hat{P}_n|^2 \, d\mu = \sum_{(n)} c^2(n_1, ..., n_k) = \mathcal{O}(\vartheta^{2n}).$$

This holds for all $\vartheta > \|C\|$ and therefore with the Cauchy–Schwarz inequality we obtain

$$\sum_{(n)} |c(n_1, ..., n_k)| = \mathcal{O}(\vartheta^n). \tag{2.1.18}$$

Let J be any non-empty subset of $\{1, ..., k\}$ and $|J|$ its size. If C_J is the submatrix of C with the row and column numbers from J then

$$|I - CY| = 1 + \sum_{r=1}^{k} D_r \tag{2.1.19}$$

with $D_r = (-1)^r \sum_{|J|=r} |C_J| \prod_{j \in J} y_j$ $(r = 1, ..., k)$. By rearranging the terms of the binomial expansion of $|I - CY|^{-p}$ we obtain

$$\hat{P}_n(y_1, ..., y_k) = (\Gamma(p))^{-1} \sum_{n_1 + 2n_2 + \cdots + kn_k = n} \Gamma(p + n_1 + \cdots + n_k)$$

$$\times \prod_{r=1}^{k} (-D_r)^{n_r}/n_r!. \tag{2.1.20}$$

If $c_{11} = \cdots = c_{kk} = 0$ then D_1 ad $\hat{P}_1$ vanish. In particular for $p = 1$ we have the recursion formula

$$\hat{P}_n = -\sum_{r=1}^{\min(k,n)} D_r \hat{P}_{n-r} \qquad (n \geqslant 1, \hat{P}_0 := 1). \tag{2.1.21}$$

For larger values of p the coefficients $c(n_1, ..., n_k)$ of the polynomials $\hat{P}_n$ are better computed recursively by multiplication of the series with smaller values of p than from (2.1.20).

2.2. *Convergence of the General Expansions*

Some of the convergence properties of the series (2.1.13) are summarized by the following theorem:

THEOREM 2.1. *Let R be a non-singular kxk-correlation matrix, $2p$ a positive integer, $W = \mathrm{Diag}(w_1, ..., w_k)$ with any positive numbers w_j, X a $\chi_k^2(2p, R)$-distributed random vector and $\mathcal{R} = \otimes_{j=1}^{k} (a_j, b_j)$ any rectangular region with $0 \leqslant a_j < b_j \leqslant \infty$; then the following statements hold*:

(a) *If* $\|I - WRW\| < 1$ *then the series* (2.1.13a) *is absolutely convergent to* $\Pr\{X \in \mathscr{R}\}$.

(b) *The series* (2.1.13b) *is always absolutely convergent to* $\Pr\{X \in \mathscr{R}\}$ *for any bounded* $\mathscr{R}$. *If* $\|I - (WRW)^{-1}\| < 1$ *this holds also for any unbounded* $\mathscr{R}$.

(c) *The series* (2.2.13c) *converges always absolutely to* $\Pr\{X \in \mathscr{R}\}$.

Remarks. With $W = I$ the sufficient condition in (a) is $\|\dot{R}\| < 1$. For $k = 3$ this is easily shown to be equivalent to (1.2) but for $k > 3$ the condition $\|\dot{R}\| < 1$ is weaker than (1.2). For example, $\|\dot{R}\| < 1$ holds for the 4×4-tridiagonal matrix R with $r_{i,i\pm 1} = 1/\sqrt{3}$ but (1.2) is not satisfied.

Especially for $p = \frac{1}{2}$ the condition $\|\dot{R}\| < 1$ is sufficient for the absolute convergence of the "tetrachoric" expansion of a multivariate normal probability of a symmetrical rectangle $\otimes_{j=1}^{k} (-\sqrt{b_j}, \sqrt{b_j})$. According to (1.6) and the remark preceding (1.7) this holds also for general rectangular regions $\otimes_{j=1}^{k} (a_j, b_j)$ $(-\infty \leqslant a_j < b_j \leqslant \infty)$, since the tetrachoric expansion is an orthogonal expansion with Hermite polynomials. The condition $\|\dot{R}\| < 1$ is more general than the criterion $\max_{i<j} |r_{ij}| < (k-1)^{-1}$ given in Theorem 2 of Harris and Soms [5]. The convergence can always be achieved by suitable scale factors d_j. For a more general "shifted" tetrachoric series see also Royen [17].

In the proof of Theorem 2.1 the following simple lemma is used:

LEMMA 2.1. *For any positive numbers p and* $x_1, ..., x_k$ *the sequences* (m_n) *and* (M_n), *defined by*

$$m_n := \left(\max \left\{ \prod_{j=1}^{k} g_{p+n_j}(x_j) \,\middle|\, \sum_j n_j = n \right\} \right)^{1/n}$$

$$M_n := \left(\max \left\{ \prod_{j=1}^{k} G_{p+n_j}(x_j) \,\middle|\, \sum_j n_j = n \right\} \right)^{1/n}$$

tend to zero for $n \to \infty$.

Proof. It is $\prod_{j=1}^{k} g_{p+n_j}(x_j) < \prod_{j=1}^{k} x_j^{p-1} \prod_{j=1}^{k} x_j^{n_j} / \prod_{j=1}^{k} \Gamma(p+n_j)$. Since $1/\Gamma(p+n)$ is a log-concave function of n, it follows that $(\prod_{j=1}^{k} x_j^{n_j} / \prod_{j=1}^{k} \Gamma(p+n_j))^{1/n} \leqslant (\max_j x_j)/(\Gamma(p+n/k))^{k/n} \simeq (\max_j x_j) ek/n$ for $n \to \infty$ and therefore $m_n \to 0$. From $\prod_{j=1}^{k} G_{p+n_j}(x_j) < \prod_{j=1}^{k} x_j^p \prod_{j=1}^{k} x_j^{n_j} / \prod_{j=1}^{k} \Gamma(p+1+n_j)$ it also follows that $M_n \to 0$. ∎

Proof of Theorem 2.1. (a) Because of (2.1.7) and (2.1.18) the series, obtained from (2.1.13a) (with $a_j = 0$, $b_j = x_j$) by termwise differentiation, has a majorant $\in \mathscr{L}^1(\mathbf{R}^k)$. Using (2.1.5) with $m = n$ termwise integration shows the coincidence of its F.t. with the expansion (2.1.12a) of $\psi_{k,d,R}$, which proves (a).

(b) The absolute convergence of the series (2.1.13b) follows from Lemma 2.1. It is sufficient to prove only the convergence to the d.f. $G_{k,d,R}$. We write the multivariate normal density with any $W = \mathrm{Diag}(w_1, ..., w_k) > 0$, $A = (WRW)^{-1}$, and $d_j = w_j^2/2$ in the following form:

$$\varphi_{k,R}(x) = (2\pi)^{-k/2}\,|A|^{1/2}\,|W| \exp(-\tfrac{1}{2}x'W(A-I)Wx)$$
$$\times \exp\left(-\sum_{j=1}^{k} d_j x_j\right).$$

After having expanded the first exponential factor into a power series we obtain by integration over $\mathscr{R} = \bigotimes_{j=1}^{k}(-\sqrt{b_j}, \sqrt{b_j})$ that

$$G_{k,1R}(b_1, ..., b_k)$$
$$= |A|^{1/2} \sum_{n=0}^{\infty} \sum_{(n)} c^*(n_1, ..., n_k; 1/2) \prod_{j=1}^{k} G_{1/2+n_j}(d_j b_j)$$

with certain coefficients satisfying $\sum_{(n)} |c^*(n_1, ..., n_k; 1/2)| = \mathcal{O}_{n\to\infty}(\gamma^n)$, where γ is any sufficiently large number. Ândel's series is the special case with $w_j = \sqrt{r^{jj}}$ $(j = 1, ..., k)$. By convolution the densities

$$g_{k,2p,R}(x_1, ..., x_k)$$
$$= |A|^p \sum_{n=0}^{\infty} \sum_{(n)} c^*(n_1, ..., n_k; p) \prod_{j=1}^{k} (d_j\, g_{p+n_j}(d_j x_j))$$

are obtained. For their Laplace transforms we find with sufficiently large arguments s_j and $z_j = (1 + s_j/d_j)^{-1}$:

$$|A|^p \left(\prod_{j=1}^{k} z_j^p\right)\left(\sum_{n=0}^{\infty} \sum_{(n)} c^*(n_1, ..., n_k; p) \prod_{j=1}^{k} z_j^{n_j}\right).$$

The series within the brackets is an analytical function of the z_j and therefore the coefficients c^* must coincide with the coefficients $c(n_1, ..., n_k)$ in (2.1.13b), which implies the first assertion of (b). If $\|C\| < 1$ then passing to the limits $b_j = \infty$ is possible, since the series (2.1.13b) converges uniformly on $[0, \infty]^k$ due to (2.1.18).

(c) Because of (2.1.11), $\|C\| < 1$, and (2.1.18) it follows that the series (2.1.13c) (with $a_j = 0$, $b_j = x_j$) is absolutely convergent to a uniformly bounded function $G^*(x_1, ..., x_k)$. For the Laplace transforms $\hat{G}^*$ and $\hat{g}_{k,2p,R}$ we obtain by termwise integration the relation $\hat{G}^*(s_1, ..., s_k) = \hat{g}_{k,2p,R}(s_1, ..., s_k)/\prod_{j=1}^{k} s_j$, which proves $G^* = G_{k,2p,R}$. ∎

Some Further Remarks. Comparing Theorem 2.1(a) with (1.4) and (1.5) we have the following identities, if $p > \frac{1}{2}$:

$$\begin{aligned} w_j(x_j) &= d_j\, g_p(d_j x_j), \\ L_{j,n_j}(x_j) &= \left|\binom{-p}{n_j}\right|^{-1/2} L_{n_j}^{(p-1)}(d_j x_j), \\ c^*(n_1, ..., n_k) &= c(n_1, ..., n_k)\left(\prod_{j=1}^{k}\left|\binom{-p}{n_j}\right|\right)^{-1/2} \\ c(n_1, ..., n_k) &= E\left(\prod_{j=1}^{k} L_{n_j}^{(p-1)}(d_j X_j)\right), \end{aligned} \tag{2.2.1}$$

where the expectation refers to the density $g_{k,2p,R}$.

Also expansions with an order $\alpha \neq p-1$ are possible. From the generating function $\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) u^n = (1-u)^{-1-\alpha} \exp(-ux/(1-u))$ ($|u| < 1$, A.S.22.9.15) it follows with u_j from (2.1.2) and $-u_j/(1-u_j) = it_j/d_j$:

$$\psi_{k,2p,R} = \psi_0^{1+\alpha}\, E\left(\sum_{n=0}^{\infty} \sum_{(n)} \prod_{j=1}^{k} L_{n_j}^{(\alpha)}(d_j X_j) u_j^{n_j}\right)$$
$$(-1/2 \leqslant \alpha \leqslant p-1). \tag{2.2.2}$$

From $\max\{|u_j|\} < 1$, the bounds $|L_{n_j}^{(\alpha)}(d_j x_j)| \leqslant |\binom{-\alpha-1}{n_j}| \exp(\frac{1}{2} d_j x_j)$ ($x \geqslant 0$, $\alpha \geqslant 0$, A.S.22.14.13), the integrability of $g_{k,2p,R} \exp(\frac{1}{2}\sum_{j=1}^{k} d_j x_j)$ according to Theorem 1.1 and (2.1.15a) follows the termwise integrability in (2.2.2), i.e.,

$$\psi_{k,2p,R} = \psi_0^{1+\alpha} \sum_{n=0}^{\infty} \sum_{(n)} c^{(\alpha)}(n_1, ..., n_k) \prod_{j=1}^{k} u_j^{n_j} \tag{2.2.3}$$

with

$$c^{(\alpha)}(n_1, ..., n_k) = E\left(\prod_{j=1}^{k} L_{n_j}^{(\alpha)}(d_j X_j)\right),$$

which are the coefficients of the expansion of $\prod_{j=1}^{k} (1-u_j)^{p-1-\alpha}$ $|I - CU|^{-p}$. This also holds for $\alpha = p-1 = -\frac{1}{2}$. By inversion of (2.2.3) it follows that

$$G_{k,2p,R}(x_1, ..., x_k) = \sum_{n=0}^{\infty} \sum_{(n)} c^{(\alpha)}(n_1, ..., n_k) \prod_{j=1}^{k} G_{1+a+n_j}^{(n_j)}(d_j x_j) \tag{2.2.4}$$

$(C = I - WRW,\ D = \mathrm{Diag}(d_1, ..., d_k) = \frac{1}{2}W^2,\ R^{-1} - D > 0,\ -\frac{1}{2} \leqslant \alpha \leqslant p-1)$.

The density $g_{k,1,R}$ follows from the multivariate normal density by differentiation of $G_{k,1,R}(x_1, ..., x_k)$. We find

$$g_{k,1,R}(x_1, ..., x_k) = (8\pi)^{-k/2} |R|^{-1/2}$$

$$\times \sum_s \exp\left(-\tfrac{1}{2} \sum_{i,j=1}^{k} r^{ij} s_i s_j \sqrt{x_i}\sqrt{x_j}\right) \Big/ \left(\prod_{j=1}^{k} x_j\right)^{1/2} \tag{2.2.5}$$

where $s = (s_1, ..., s_k)$ assumes all possible values $(\pm 1, ..., \pm 1)$. For $2p > 1$ absolutely convergent series for the densities $g_{k,2p,R}$ are obtained by termwise differentiation in (2.1.13). For this only in (2.1.13a) we assume $\|I - WRW\| < 1$.

A generalization of the expansions (2.1.13) and of Theorem 2.1 to multivariate gamma distributions with non-integer values of $2p$ is possible but the function $\psi_{k,2p,R}$ in (2.1.1) is not for all correlation matrices and all positive values of p a ch.f. of a d.f. (Griffiths [4]).

2.3. *Expansions with Special Scale Factors*

We might consider the scale factors w_j as "natural" ones if they let all the diagonal elements of C vanish. In (2.1.12a) these are the factors $w_j = 1$ and in (2.1.12b) the factors $w_j = \sqrt{r^{jj}}$. Also in (2.1.12c) such a choice of the scale factors is always possible. This follows from Lemma 2.2 with $w_j = x_j^{-1/2}$.

LEMMA 2.2. *Let R be a non-singular kxk-correlation matrix and $x = (x_1, ..., x_k)'$ be any vector with positive components. Furthermore, let $A(x) = (a_{ij})$ be the matrix $R + X$ with $X = \mathrm{Diag}(x_1, ..., x_k)$ and (a^{ij}) the inverse of A. There exists exactly one solution x of the equations*

$$d_i(x) := 2x_i a^{ii}(x) = 1 \qquad (i = 1, ..., k). \tag{2.3.1}$$

Proof. The function $f(x)$ with the components $f_i(x) = 1/(2a^{ii}) = x_i/d_i$ has the partial derivatives $\partial f_i/\partial x_j = (a^{ij}/a^{ii})^2/2$ $(i, j = 1, ..., k)$. With $a^{ii} = |(R+X)_{ii}|/|R+X| \geqslant 1/(1+x_i)$ we find for $x^{(0)} := (1, ..., 1)'$ that $d_i(x^{(0)}) \geqslant 1$ and therefore $f(x^{(0)}) \leqslant x^{(0)}$. For the sequence $(x^{(N)})$, defined by $x^{(N)} = f(x^{(N-1)})$, it follows that $x^{(N+1)} - x^{(N)} \leqslant 0$ and consequently $d_i(x^{(N)}) \geqslant 1$ for all N.

Here "$x \leqslant y$" stands for $x_i \leqslant y_i$ $(i = 1, ..., k)$. If $x_i < y_i$ occurs at least for one index i then also $x < y$ is written. Finally "$x \ll y$" means $x_i < y_i$ for all i. The sequence $(x^{(N)})$ has a limit $x^* \gg 0$, which is a solution of (2.3.1).

Uniqueness of the solution. For $k > 2$ the function f is not globally contractive so the fixed point theorem cannot be applied directly.

Let t be the trace $\sum_{j=1}^{k} d_j(x) = 2\sum_{j=1}^{k} x_j a^{jj}$ with the derivatives

$\frac{1}{2}(\partial t/\partial x_i) = a^{ii} - \sum_{j=1}^{k} x_j a^{ij2}$. The matrix $B = X^{1/2} A^{-1} X^{1/2}$ has only characteristic roots $\beta_j \in (0, 1)$. Let (z_{ij}) be an orthogonal matrix with the eigenvectors of B as columns. For the ith diagonal element of B^2 we find

$$\sum_{j=1}^{k} x_i a^{ij2} x_j = \sum_{j=1}^{k} (\sqrt{x_i}\, a^{ij} \sqrt{x_j})^2$$
$$= \sum_{j=1}^{k} \beta_j^2 z_{ij}^2 < \sum_{j=1}^{k} \beta_j z_{ij}^2 = x_i a^{ii},$$

which implies $\partial t/\partial x_i > 0$ $(i = 1, ..., k)$. Therefore no further solution x exists with $x > x^*$ or $x < x^*$.

Now let x be any vector with $x \geqslant 0$ and $J := \{j \mid x_j > x_j^*\} \neq \varnothing$. Again for $t_J := \sum_{j \in J} d_j(x)$ the inequalities $\partial t_J/\partial x_j > 0$ $(j \in J)$ can be shown as above. For the point y with the components

$$y_j = \begin{Bmatrix} x_j, & j \in J \\ x_j^*, & j \notin J \end{Bmatrix}$$

we find at least one $j \in J$ with $d_j(y) > 1$ and consequently $f_j(y) < x_j$. Along the straight line from y to x the component f_j is not increased and we also obtain $f_j(x) < x_j$. Therefore x is no solution of (2.3.1). ▌

For the one-factorial correlation matrices a different choice of the factors w_j leads to simpler results. With

$$R := \begin{cases} W^{-2} + aa', & w_j^2 = 1/(1 - a_j^2), \quad -1 < a_j < 1 & \text{(2.3.2a)} \\ W^{-2} - aa', & w_j^2 = 1/(1 + a_j^2), \quad 1 - \sum_{j=1}^{k} (a_j w_j)^2 > 0 & \text{(2.3.2b)} \end{cases}$$

and $d_j = w_j^2/2$ it follows for the matrices C in (2.1.12) that

$$C = (2\lambda)^{-1} \beta W a a' W, \qquad |I - CY| = 1 - \frac{\beta}{\lambda} \sum_{j=1}^{k} a_j^2 d_j y_j \tag{2.3.3}$$

with

$$y_j = u_j, \qquad \lambda = 1, \qquad \beta = \mp 2 \tag{2.3.3a}$$
$$y_j = z_j, \qquad \lambda = 1 + \beta \sum_{j=1}^{k} a_j^2 d_j, \qquad \beta = \pm 2 \tag{2.3.3b}$$
$$y_j = \omega_j, \qquad \lambda = 1 + \beta \sum_{j=1}^{k} a_j^2 d_j, \qquad \beta = \pm 1. \tag{2.3.3c}$$

Here the upper sign holds for (2.3.2a) and the lower one for (2.3.2b).

Now we obtain with (2.1.12), (2.1.13), and (2.3.3) the series

$$G_{k,2p,R}(x_1, \ldots, x_k) = (\lambda^p \Gamma(p))^{-1} \sum_{n=0}^{\infty} \Gamma(p+n)\lambda^{-n}$$
$$\times \sum_{(n)} \prod_{j=1}^{k} F_{p,n_j}(d_j x_j)(\beta a_j^2 d_j)^{n_j}/n_j!. \quad (2.3.4)$$

The only additional assumption for the absolute convergence of these series is $\|C\| = \sum_{j=1}^{k} a_j^2 w_j^2 < 1$ in case of (2.3.3a) with (2.3.2a). Also some integral representations follow from (2.3.4). With (2.3.3b) and $G_p(x, y) := e^{-y} \sum_{n=0}^{\infty} G_{p+n}(x)\, y^n/n!$ we find

$$G_{k,2p,R}(x_1, \ldots, x_k) = \int_0^{\infty} \left(\prod_{j=1}^{k} G_p(d_j x_j, \pm 2a_j^2 d_j y) \right) g_p(y)\, dy, \quad (2.3.5)$$

which also holds for a singular R in (2.3.2b) with $\lambda = 1 - \sum_{j=1}^{k} a_j^2 w_j^2 = 0$ (Royen [16, 18]).

3. Some Formulas for the Three-Variate Chi-Square Distribution

In this section the off-diagonal elements of any symmetrical 3×3-matrix (b_{ij}) are also indexed by single indices, i.e., $b_1 = b_{23}$, $b_2 = b_{13}$, $b_3 = b_{12}$. For $k = 3$ the unique solutions $x_j = w_j^{-2}$ of (2.3.1) in Lemma 2.2 can be computed by elimination of x_2 and x_3. With $|R_{ii}| = 1 - r_i^2$ the solution x_1 is the unique solution of

$$\frac{x^2 - |R_{22}|\,|R_{33}|}{\sqrt{(x + |R_{22}|)(x + |R_{33}|)}} = \frac{|R| - |R_{11}|\, x^2}{\sqrt{(1+x)(|R| + |R_{11}|\, x)}} \quad (3.1)$$

between the bounds $\sqrt{|R_{22}|\,|R_{33}|}$, $\sqrt{|R|/|R_{11}|} \leqslant 1$. With $v = (x_1 + |R_{22}|)/(x_1 + |R_{33}|)$ the two remaining solutions are

$$x_2 = ((|R| + |R_{11}|\, x_1)/(v(1 + x_1)))^{1/2}, \qquad x_3 = v x_2. \quad (3.2)$$

From (2.1.19) and (2.1.20) with $k = 3$ and $N \geqslant 2$, we obtain

$$\hat{P}_N(y_1, y_2, y_3)$$
$$= \frac{1}{\Gamma(p)} \sum_{2n_2 + 3n_3 = N} \Gamma(p + n_2 + n_3) \frac{(c_1^2 y_2 y_3 + c_2^2 y_1 y_3 + c_3^2 y_1 y_2)^{n_2}}{n_2!}$$
$$\times \frac{(2c_1 c_2 c_3 y_1 y_2 y_3)^{n_3}}{n_3!}$$
$$= \frac{1}{\Gamma(p)} \sum_{2n_2 + 3n_3 = N} \frac{2^{n_3}}{n_3!} \Gamma(p + (N - n_3)/2)$$
$$\times \sum_{l_1 + l_2 + l_3 = n_2} \prod_{j=1}^{3} c_j^{2l_j + n_3} y_j^{n_2 + n_3 - l_j}/l_j!. \quad (3.3)$$

With $M_j := 2l_j + n_3$ and $2(n_2 + n_3 - l_j) = N - n_3 - (M_j - n_3) = N - M_j$ it follows that

$$\hat{P}_N = \frac{1}{\Gamma(p)} \sum_{M_1 + M_2 + M_3 = N} \sum_{0 \leqslant n_3 \leqslant \min M_j} \frac{2^{n_3}}{n_3!} \times \frac{\Gamma(p + (N - n_3)/2)}{\prod_{j=1}^{3} ((M_j - n_3)/2)!} \prod_{j=1}^{3} c_j^{M_j} y_j^{(N - M_j)/2},$$

where n_3, M_1, M_2, M_3, and N have identical parity. With $M_j = 2mj$, $n_3 = 2m$, $N = 2n$ or $M_j = 2m_j + 1$, $n_3 = 2m + 1$, $N = 2n + 1$, we obtain by inversion the d.f.

$$G_{3,2p,R}(x_1, x_2, x_3) = c\left(\prod_{j=1}^{3} G_p(d_j x_j) + \sum_{N=2}^{\infty} P_N(x_1, x_2, x_3)\right) \tag{3.4}$$

with

$$P_N(x_1, x_2, x_3) = \begin{cases} \dfrac{1}{\Gamma(p)} \displaystyle\sum_{m_1 + m_2 + m_3 = n} \left(\sum_{m=0}^{\min m_j} \frac{2^{2m}}{(2m)!} \frac{\Gamma(p + n - m)}{\prod_{j=1}^{3} (m_j - m)!}\right) \\ \qquad \times \displaystyle\prod_{j=1}^{3} c_j^{2m_j} F_{p,n-m_j}(d_j x_j), \qquad (N = 2n, n \geqslant 1) \\ \dfrac{1}{\Gamma(p)} \displaystyle\sum_{m_1 + m_2 + m_3 = n-1} \left(\sum_{m=0}^{\min m_j} \frac{2^{2m+1}}{(2m+1)!} \frac{\Gamma(p + n - m)}{\prod_{j=1}^{3} (m_j - m)!}\right) \\ \qquad \times \displaystyle\prod_{j=1}^{3} c_j^{2m_j+1} F_{p,n-m_j}(d_j x_j), \qquad (N = 2n + 1, n \geqslant 1) \end{cases}$$

and

$$c = 1, \qquad d_j = 1/2, \qquad c_j = -r_j, \qquad F_{p,n-m} = G_{p+n-m}^{(n-m)}, \qquad \|\dot{R}\| < 1 \tag{3.5a}$$

$$\begin{gathered} c = |Q|^p, \qquad Q = (q_{ij}) \qquad \text{with} \quad q_{ij} = r^{ij}/(r^{ii} r^{jj})^{1/2}, \qquad d_j = r^{jj}/2, \\ c_j = -q_j = -q_{il}, \qquad F_{p,n-m} = G_{p+n-m} \end{gathered} \tag{3.5b}$$

$$\begin{gathered} c,\ C = (c_{ij}) \qquad \text{from (2.1.12c) with} \quad c_{jj} = 0, \\ d_j = w_j^2/2 \quad \text{following from (3.1), (3.2),} \qquad F_{p,n-m} = H_{p,n-m}. \end{gathered} \tag{3.5c}$$

For a one-factorial matrix R the series (2.3.4) is simpler. In a certain sense each 3×3-correlation matrix R with $r_1 r_2 r_3 \neq 0$ is "one-factorial." With $s_i = \text{sgn}(r_i)$, $s = \text{sgn}(r_1 r_2 r_3)$, and $a_i = s_i\, |r_j r_l / r_i|^{1/2}$ (i, j, l any permutation of 1, 2, 3), we have

$$r_l = r_{ij} = s a_i a_j \qquad (i \neq j). \tag{3.6}$$

If $s=1$ and $a^2=\max\{a_j^2\}>1$ then a formal calculation starting from (2.1.12) with one negative factor $w^2=(1-a^2)^{-1}$ does not lead to the correct probability. But for these cases an expansion of the type (2.1.13c) can be derived, which is simpler than (3.5c). We assume without loss of generality that

$$r_3^2=\min\{r_j^2\} \qquad \text{and} \qquad a_3^2=r_1r_2/r_3>1. \tag{3.7}$$

With

$$w_j=|1-a_j^2|^{-1/2}, \tag{3.8}$$

we obtain the following elements of $C=I-2(I+WRW)^{-1}$:

$$\begin{aligned} &c_{11}=c_{22}=c_{12}=0, && c_{33}=1-(w_1^2+w_2^2)/(a_3^2w_3^2) \\ &c_{13}=(w_1r_3)/(w_3r_1), && c_{23}=(w_2r_3)/(w_3r_2). \end{aligned} \tag{3.9}$$

With

$$|I-C\Omega|=1-\omega_3(c_{33}+c_{13}^2\omega_1+c_{23}^2\omega_2), \tag{3.10}$$

$$c=8^p\,|I+WRW|^{-p}=2^p(1-a_3^{-2})^p, \tag{3.11}$$

and (3.7) it follows that

$$\begin{aligned} &G_{3,2p,R}(x_1,x_2,x_3) \\ &=\frac{c}{\Gamma(p)}\sum_{N=0}^{\infty}\sum_{\substack{n_1+n_2+n_3=N\\ n_1+n_2\leqslant n_3}}\Gamma(p+n_3) \\ &\quad\times\frac{c_{13}^{2n_1}c_{23}^{2n_2}c_{33}^{n_3-n_1-n_2}}{n_1!\,n_2!(n_3-n_1-n_2)!}\prod_{j=1}^{3}H_{p,n_j}\left(\frac{1}{2}w_j^2x_j\right). \end{aligned} \tag{3.12}$$

If $\max\{a_j^2\}=a_3^2=1$ then with $a_3=1$, $a_1=r_2$, $a_2=r_1$, and

$$w_j^2:=r^{jj}=(1-a_j^2)^{-1}\quad(j=1,2),\qquad w_3^2:=r^{33}=(1-a_1^2a_2^2)w_1^2w_2^2,$$
$$r^{12}=0,\qquad r^{j3}=-a_jw_j^2\quad(j=1,2),$$

we have the standardized inverse $Q=(WRW)^{-1}$ with the off-diagonal elements

$$q_{12}=0,\qquad q_{j3}=-a_jw_j/\sqrt{\lambda'}\quad(j=1,2),$$
$$\lambda'=1+\sum_{j=1}^{2}a_j^2w_j^2=r^{33}=|Q|^{-1}.$$

Now from (2.1.12b) with $|I-CZ| = 1 - z_3(q_{13}^2 z_1 + q_{23}^2 z_2)$ we obtain the d.f.

$$
\begin{aligned}
G_{3,2p,R}&(x_1, x_2, x_3) \\
&= (\lambda'^p \Gamma(p))^{-1} \sum_{n=0}^{\infty} \Gamma(p+n) G_{p+n}(r^{33} x_3/2) \\
&\quad \times \sum_{(n)} \prod_{j=1}^{2} (q_{j3}^{2n_j}/n_j!) G_{p+n_j}(r^{jj} x_j/2) \\
&= \int_0^{x_3/2} \left(\prod_{j=1}^{2} G_p(r^{jj} x_j/2, a_j^2 r^{jj} x) \right) g_p(x)\, dx.
\end{aligned}
$$

Besides, we have $\|C\| = \|\dot{Q}\| = (1-\lambda'^{-1})^{1/2} < 1$. The integral can also be derived from (2.3.5) by the passage to the limit $\lim_{a_3 \to 1_-} \partial G_{3,2p,R}/\partial x_3$.

Finally, if, e.g., $r_{12} = 0$ then $|R| = 1 - r_{13}^2 - r_{23}^2 > 0$ implies $\|\dot{R}\| = (r_{13}^2 + r_{23}^2)^{1/2} < 1$ and it follows from (2.1.12a) that

$$
\begin{aligned}
G_{3,2p,R}(x_1, x_2, x_3) &= (\Gamma(p))^{-1} \sum_{n=0}^{\infty} \Gamma(p+n) G_{p+n}^{(n)}(x_3/2) \\
&\quad \times \sum_{(n)} \prod_{j=1}^{2} (r_{j3}^{2n_j}/n_j!) G_{p+n_j}^{(n_j)}(x_j/2).
\end{aligned} \tag{3.14}
$$

If R is singular with correlations $|r_j| < 1$ then $\operatorname{rank}(R) = 2$. If $(S_{11}, S_{22}, S_{12})'$ has a $W_2(d, R_{33})$-density (i.e., a Wishart-density with $d = 2p \geqslant 2$) then $g_{3,d,R}$ can be computed by the linear transformation $X_j = S_{jj}$ $(j = 1, 2)$, $X_3 = b_1 S_{11} + b_2 S_{22} + 2b_1 b_2 S_{12}$ with $b_j = (r_i - r_j r_3)/(1 - r_3^2)$ $(i, j \leqslant 2,\ i \neq j)$. Let $Q = (q_{ij})$ be the 3×3-matrix with the elements $q_{ij} = q_l = (1 - r_l)^{-1}$ $(i \neq j)$ and $q_{ii} = -q_j q_l / q_i$ (i, j, l any permutation of 1, 2, 3). With the cone $\mathscr{C} := \{x \in \mathbf{R}_+ \mid x'Qx > 0\}$ and the vector v with the components $v_i = \frac{1}{2} r_i/(r_i - r_j r_l)$, the result is

$$
g_{3,d,R}(x) = \begin{cases} \dfrac{(x'Qx)^{(d-3)/2} \exp(-v'x)}{\pi 2^d \Gamma(d-1) (\prod_{j=1}^{3} (1 - r_j^2))^{1/2}}, & x \in \mathscr{C} \\ 0, & x \notin \mathscr{C}. \end{cases} \tag{3.15}
$$

The integration of $q_{3,d,R}$ over the intersection of $\mathscr{C}$ with any rectangular region is tedious. If $r_1, r_2, r_3 < 0$ then the integral in (2.3.5) with $k = 3$ and $v_j = d_j = \frac{1}{2}(1 + a_j^2)^{-1}$ $(a_j^2 = -r_i r_l / r_j)$ can be used for all $d \geqslant 1$. Obviously we have $\exp(\frac{1}{2} v'x)\, g_{3,d,R}(x) \in \mathscr{L}^1 \cap \mathscr{L}^2$ for $d \geqslant 3$. Thus also the expansion (2.2.3) can be applied with $\alpha = 0$ or $\alpha = \frac{1}{2}$. This holds also for

$\|C\| = \|I - WRW\| = 1$, since $\|U\| < 1$. For any region $\mathscr{A} \subseteq \mathbf{R}^3_+$ it follows from (1.6) that

$$\Pr\{X \in \mathscr{A}\} = \sum_{n=0}^{\infty} \sum_{(n)} c^{(\alpha)}(n_1, n_2, n_3) \times \int_{\mathscr{A}} \left(\prod_{j=1}^{3} d_j g_{1+\alpha+n_j}^{(n_j)}(d_j x_j) \right) d\mu, \tag{3.16}$$

where the coefficients are defined by

$$\sum_{n=0}^{\infty} \sum_{(n)} c^{(\alpha)}(n_1, n_2, n_3) \prod_{j=1}^{3} u_j^{n_j} = \left(\prod_{j=1}^{3} (1-u_j)^{p-1-\alpha} \right) \times \left((\Gamma(p))^{-1} \sum_{n=0}^{\infty} \Gamma(p+n) \left(\sum_{j=1}^{3} 2a_j^2 d_j u_j \right)^n \Big/ n! \right). \tag{3.17}$$

Also for a singular R with $r_1 r_2 r_3 > 0$ a similar $\mathscr{L}^2$-argument can be used if $d \geqslant 3$. With (2.1.3), (2.1.4), $\psi_0 = \prod_{j=1}^3 (1 - it_j/d_j)^{-1} = \prod_{j=1}^3 ((1+\omega_j)/2)$ and with (3.10), (3.11) we write the ch.f. ψ as $\psi_0 \psi^*$, where ψ^* has the formal expansion

$$\prod_{j=1}^{3} ((1+\omega_j)/2)^{p-1} \left(\frac{c}{\Gamma(p)} \sum_{n=0}^{\infty} \Gamma(p+n) \omega_3^n (c_{33} + c_{13}^2 \omega_1 + c_{23}^2 \omega_2)^n / n! \right) = \sum_{n=0}^{\infty} \sum_{(n)} c^*(n_1, n_2, n_3) \prod_{j=1}^{3} \omega_j^{n_j}. \tag{3.18}$$

THEOREM 3.1. *Let X be a $\chi_3^2(2p, R)$-distributed random vector with $p > 1$ and R a singular 3×3-correlation matrix of rank 2 with a positive product of its off-diagonal elements r_j and $r_3^2 = \min\{r_j^2\}$. With the numbers $d_i = \frac{1}{2} |1 - r_j r_l / r_i|^{-1}$ (i, j, l any permutation of 1, 2, 3) and the coefficients c^* defined in (3.18), the expansion*

$$\Pr\{X \in \mathscr{A}\} = \sum_{n=0}^{\infty} \sum_{(n)} c^*(n_1, n_2, n_3) \int_{\mathscr{A}} \left(\prod_{i=1}^{3} d_i h_{1,n_i}(d_i x_i) \right) d\mu$$

holds at least for any bounded region $\mathscr{A} \subseteq \mathbf{R}^3_+$.

Proof. With $d\varphi_j/dt_j = 2(1 + t_j^2/d_j^2)^{-1}/d_j$ we have

$$\int_{(-\pi,\pi)^3} |\psi^*|^2 \, d\mu(\varphi) = 8 \int_{\mathbf{R}^3} |\psi_{3,2p,R}|^2 \, d\mu(t) < \infty,$$

since $g_{3,2p,R} \in \mathscr{L}^2(\mathbf{R}^3)$ for $p > 1$.

With the w_j from (3.8) and $|R| = 0$ it follows from (3.10) that

$$|I + C| = |2(I - (I + WRW)^{-1})| = 1 + c_{33} - c_{13}^2 - c_{23}^2 = 0$$

and, since $\psi_{3,2p,R}(0) = 1$, it follows from (3.11) that

$$|I - C| = 1 - c_{33} - c_{13}^2 - c_{23}^2 = 2(1 - r_3/(r_1 r_2)).$$

Therefore we obtain $c_{13}^2 + c_{23}^2 = r_3/(r_1 r_2) < 1$ and $c_{33} < 0$. Since $\omega_j = e^{i\varphi_j} \neq -1$ it follows that $|c_{33} + c_{13}^2 \omega_1 + c_{23}^2 \omega_2| < 1$.

The power series for $(1 + \omega_j)^{p-1}$ is absolutely convergent on the unit circle since $p > 1$. Writing at first (3.18) as an absolutely convergent power series of ω_3 the coefficients $c^*(n_1, n_2, n_3)$ are easily seen to be the Fourier coefficients of $\psi^*(\varphi_1, \varphi_2, \varphi_3)$.

Now the proof is completed by a conclusion similar to (1.6) but $\mathscr{A}$ is supposed to be bounded since the weight function w in (1.6) is replaced here by $w \equiv 1$.[1] ∎

In some multiple test procedures, based on union intersection tests, Bonferroni inequalities of higher order are useful for small values of the tail probabilities $\Pr\{\max_j X_j > x\}$. With the events $A_j = \{X_j \leqslant x\}$, $\bar{A}_j = \{X_j > x\}$ $(j = 1, ..., k)$ and the sums

$$S_m = \sum_{|J| = m} P_J, \qquad P_J = \Pr\left(\bigcap_{j \in J} A_j\right)$$

$$\bar{S}_m = \sum_{|J| = m} \bar{P}_J, \qquad \bar{P}_J = \Pr\left(\bigcap_{j \in J} \bar{A}_j\right);$$

e.g., the Bonferroni inequality

$$p := \Pr\left(\bigcup_{j=1}^{k} \bar{A}_j\right) \leqslant \sum_{\mu=1}^{r} (-1)^{\mu-1} \bar{S}_\mu$$

of an odd order r can be applied with $r = 3$ also to a $\chi_k^2(d, R)$-distributed random vector X with a singular correlation matrix R. In particular, this has been done for the squared multivariate range of k independent normally distributed points (Royen [18]).

Here it is not necessary to compute the probabilities of any unbounded regions, since the sums $\bar{S}_m$ can also be expressed by the S_m, which leads to

$$\bar{S}_m = \sum_{\mu=1}^{m} (-1)^\mu \binom{k-\mu}{m-\mu} S_\mu \qquad (S_0 := 1).$$

[1] From (A.S.22.5.54), (A.S.13.5.14) it follows that $H_{p,n}$ and $G_{p+n}^{(n)}$ are $\mathcal{O}(n^{-p/2-1/4})$, which implies the absolute convergence of the series (3.12), (3.14) if $\operatorname{rank}(R) = 2$. Also the series (2.3.4) with (2.3.3.a), (2.3.3.c) remain convergent if $\sum a_j^2 w_j^2 = 1$ in (2.3.2b).

Especially for $r=3$ and $k \geqslant 3$ we have

$$p \leqslant \bar{S}_1 - \bar{S}_2 + \bar{S}_3 = 1 + \binom{k-1}{3} - \binom{k-2}{2} S_1 + (k-3) S_2 - S_3.$$

For some refinements of Bonferroni inequalities see Kounias and Sotirakoglou [8].

References

[1] Abramowitz, M., and Stegun, I. A. (1968). *Handbook of Mathematical Functions.* Dover, New York.

[2] Ǎndel, J. (1971). On multiple probabilities of rectangles. *Apl. Mat.* **16** 172–181.

[3] Blumenson, L. E., and Miller, K. S. (1963). Properties of generalized Rayleigh distributions. *Ann. Math. Statist.* **34** 903–910.

[4] Griffiths, R. C. (1984). Characterization of infinitely divisible multivariate gamma distributions. *J. Multivariate Anal.* **15** 13–20.

[5] Harris, B., and Soms, A. P. (1980). The use of the tetrachoric series for evaluating multivariate normal probabilities. *J. Multivariate Anal.* **10** 252–267.

[6] Jensen, D. R. (1970). A generalization of the multivariate Rayleigh distribution. *Sankhyā Ser. A* **32** 193–206.

[7] Johnson, N. L., and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions.* Wiley, New York.

[8] Kounias, S., and Sotirakoglou, K. (1989). Bonferroni bounds revisited. *J. Appl. Probab.* **26** 233–241.

[9] Krishnaiah, P. R. (1979). Some developments on simultaneous test procedures. In *Developments in Statistics*, Vol. 2, pp. 157–201. Academic Press, New York.

[10] Krishnaiah, P. R. (1980). Computations of some multivariate distributions. In *Handbook of Statistics*, Vol. 1, pp. 745–791. North-Holland, Amsterdam.

[11] Krishnaiah, P. R., and Rao, M. M. (1961). Remarks on a multivariate gamma distribution. *Amer. Math. Monthly* **68** 342–346.

[12] Krishnamoorthy, A. S., and Parthasarathy, M. (1951). A multivariate gamma type distribution. *Ann. Math. Statist.* **22** 549–557. Correction (1960). *Ann. Math. Statist.* **31** 229.

[13] Lancaster, H. O. (1963). Correlations and canonical forms of bivariate distributions. *Ann. Math. Statist.* **34** 532–538.

[14] Miller, K. S., Bernstein, R. I., and Blumenson, L. E. (1958). Generalized Rayleigh processes. *Quart. Appl. Math.* **16** 137–145.

[15] Moran, P. A. P. (1983). A new expansion for the multivariate normal distribution. *Austral. J. Statist.* **25** 339–344.

[16] Royen, T. (1984). Multiple comparisons of polynomial distributions. *Biometrical J.* **26** 319–332.

[17] Royen, T. (1987). An approximation for multivariate normal probabilities of rectangular regions. *Statistics* **18** 389–400.

[18] Royen, T. (1991). Multivariate gamma distributions with one-factorial accompanying correlation matrices and applications to the distribution of the multivariate range. *Metrika* **38**.

[19] Šidák, Z. (1971). Remarks on Anděl's paper "On Multiple Normal Probabilities of Rectangles," *Apl. Mat.* **19** 182–187.